Multibracket simple Lie algebras *

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Abstract

We introduce higher-order (or multibracket) simple Lie algebras that generalize the ordinary Lie algebras. Their 'structure constants' are given by Lie algebra cohomology cocycles which, by virtue of being such, satisfy a suitable generalization of the Jacobi identity. Finally, we introduce a nilpotent, complete BRST operator associated with the l multibracket algebras which are based on a given simple Lie algebra of rank l.

Given $[X,Y] := XY - YX$, the standard Jacobi identity (JI) $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$ is automatically satisfied if the product is associative. For a Lie algebra $\mathcal{G}$, $[X_i,X_j]=C_{ij}^k X_k$, the JI may be written in terms of C_{ij}^k as

$$\frac{1}{2}\epsilon^{j_1 j_2 j_3}_{i_1 i_2 i_3} C^{\rho}_{j_1 j_2} C^{\sigma}_{\rho j_3} = 0 \quad . \tag{1}$$

Let $\mathcal{G}$ be simple and (for simplicity) compact. Then, the Killing metric k, with coordinates $k_{ij}=k(X_i,X_j)$, is non-degenerate and, after suitable normalization, can be brought to the form $k_{ij}=\delta_{ij}$. Moreover, k is an invariant polynomial, *i.e.*

$$k([Y,X],Z)+k(X,[Y,Z])=0 \quad . \tag{2}$$

We also know that k defines the second order Casimir invariant. Using this symmetric polynomial we may always construct a non-trivial three-cocycle

$$\omega_{i_1 i_2 i_3} := k([X_{i_1},X_{i_2}],X_{i_3}) = C^{\rho}_{i_1 i_2} k_{\rho i_3} \tag{3}$$

which is indeed skew-symmetric as consequence of (1) or (2).

In fact, it is known since the classical work of Cartan, Pontrjagin, Hopf and others that, from a topological point of view, the group manifolds of all simple compact groups are essentially equivalent to (have the [real] homology of) products of odd

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spheres, that S^3 is always present in these products and that the simple Lie algebra cocycles are, via the 'localization' process, in one-to-one correspondence with bi-invariant de Rham cocycles on the associated compact group manifolds G. This is due to the intimate relation between the order of the l(=rank $\mathcal{G}$) primitive symmetric polynomials which can be defined on a simple Lie algebra, their l associated generalized Casimir-Racah invariants [1] and the topology of the associated simple groups. Such a relation was a key fact in the eighties for the understanding of non-abelian anomalies in gauge theories [2].

The simplest (of order 3) higher-order invariant polynomial $d_{ijk} = d(X_i, X_j, X_k)$ appears for $su(3)$ (and only for A_l-type algebras, $l \geq 2$); it is given by the symmetric trace of three $su(3)$ generators. From d_{ijk} we may construct

$$\omega_{i_1 i_2 i_3 i_4 i_5} := \epsilon^{j_2 j_3 j_4}_{i_2 i_3 i_4} d([X_{i_1}, X_{j_2}], [X_{j_3}, X_{j_4}], X_{i_5}) = \epsilon^{j_2 j_3 j_4}_{i_2 i_3 i_4} C^{\rho}_{i_1 j_2} C^{\sigma}_{j_3 j_4} d_{\rho \sigma i_5} \tag{4}$$

(cf. (3)), and it can be checked that (4) defines a fifth-order invariant form (the proof will be given in the general case). The existence of this five-form ω shows us that $su(3)$ is, from a topological point of view, equivalent to $S^3 \times S^5$. If we calculate in $su(3)$ the 'four-bracket' we find that

$$[X_{j_1}, X_{j_2}, X_{j_3}, X_{j_4}] = \sum_{s \in S_4} \pi(s) X_{s(j_1)} X_{s(j_2)} X_{s(j_3)} X_{s(j_4)} = \omega_{j_1 j_2 j_3 j_4}{}^{\sigma} X_{\sigma} \quad , \tag{5}$$

where the generators X_i may be taken proportional to the Gell-Mann matrices, $X_i = \frac{\lambda_i}{2}$, and $\pi(s)$ is the parity sign of the permutation s. Thus, $\omega_{j_1 j_2 j_3 j_4}{}^{\sigma}$ is related to the four-bracket and a five-cocycle (five-form) in the same way as $C_{j_1 j_2}{}^{\sigma}$ is associated with the standard Lie bracket and a three-cocycle (three-form).

We may ask ourselves whether this construction could be extended to all the higher-order polynomials to define from them higher-order simple Lie algebras satisfying an appropriate generalization of the JI. The affirmative answer is given in [3]; we outline below the main steps that led to it. It is interesting to note that this construction may be used to produce examples of a generalization [4] of the Poisson structure different from that underlying Nambu mechanics [5].

a) Invariant polynomials on the Lie algebra $\mathcal{G}$

Let T_i be the elements of a representation of $\mathcal{G}$. Then the symmetric trace $k_{i_1 \ldots i_m} \equiv \mathrm{sTr}(T_{i_1} \ldots T_{i_m})$ (we shall only consider here sTr although not all invariant polynomials are of this form [1]; see [3]) verifies the invariance condition

$$\sum_{s=1}^{m} C^{\rho}_{\nu i_s} k_{i_1 \ldots i_{s-1} \rho i_{s+1} \ldots i_m} = 0 \quad . \tag{6}$$

Proof: By definition of k, the *l.h.s.* of (6) (cf. (2)) is

$$\mathrm{sTr}\left(\sum_{s=1}^{m} T_{i_1} \ldots T_{i_{s-1}} [T_{\nu}, T_{i_s}] T_{i_{s+1}} \ldots T_{i_m} \right) = \mathrm{sTr}\left(T_{\nu} T_{i_1} \ldots T_{i_m} - T_{i_1} \ldots T_{i_m} T_{\nu} \right) = 0 \ , \tag{7}$$

q.e.d. The above symmetric polynomial is associated to an invariant symmetric tensor field on the group G associated with $\mathcal{G}$, $k(g) = k_{i_1 \dots i_m} \omega^{i_1}(g) \otimes \dots \otimes \omega^{i_m}(g)$, where the $\omega^i(g)$ are left invariant one-forms on G. Since the Lie derivative of ω^k is given by $L_{X_i} \omega^k = -C_{ij}^k \omega^j$ for a LI vector field X_i on G, the invariance condition is the statement

$$(L_{X_\nu} k)(X_{i_1}, \dots, X_{i_m}) = -\sum_{s=1}^{m} k(X_{i_1}, \dots, [X_\nu, X_{i_s}], \dots, X_{i_m}) = 0 \tag{8}$$

c.f. (2). On forms, the invariance condition (8) may be written as

$$\epsilon_{i_1 \dots i_q}^{j_1 \dots j_q} C_{\nu j_1}^{\rho} \omega_{\rho j_2 \dots j_q} = 0 \quad . \tag{9}$$

b) Invariant forms on the Lie group G

Let $k_{i_1 \dots i_m}$ be an invariant symmetric polynomial on $\mathcal{G}$ and let us define

$$\tilde{\omega}_{\rho j_2 \dots j_{2m-2} \sigma} := k_{i_1 \dots i_{m-1} \sigma} C_{\rho j_2}^{i_1} \dots C_{j_{2m-3} j_{2m-2}}^{i_{m-1}} \quad . \tag{10}$$

Then the odd order $(2m-1)$-tensor

$$\omega_{\rho l_2 \dots l_{2m-2} \sigma} := \epsilon_{l_2 \dots l_{2m-2}}^{j_2 \dots j_{2m-2}} \tilde{\omega}_{\rho j_2 \dots j_{2m-2} \sigma} \tag{11}$$

is a fully skew-symmetric tensor. We refer to Lemma 8.1 in [4] for the proof.

Moreover, ω is an invariant form because for $q = 2m-1$ the *l.h.s.* of (9) is

$$\begin{aligned} &\epsilon_{i_1 \dots i_{2m-1}}^{j_1 \dots j_{2m-1}} C_{\nu j_1}^{\rho} \omega_{j_2 \dots j_{2m-1} \rho} = \epsilon_{i_1 \dots i_{2m-1}}^{j_1 \dots j_{2m-1}} C_{\nu j_1}^{\rho} \epsilon_{j_3 \dots j_{2m-1}}^{l_3 \dots l_{2m-1}} \tilde{\omega}_{j_2 l_3 \dots l_{2m-1} \rho} \\ &= (2m-3)! \epsilon_{i_1 \dots i_{2m-1}}^{j_1 \dots j_{2m-1}} k_{l_1 \dots l_m} C_{\nu j_1}^{l_1} \dots C_{j_{2m-2} j_{2m-1}}^{l_m} \\ &= (2m-3)! \epsilon_{i_1 \dots i_{2m-1}}^{j_1 \dots j_{2m-1}} \left[\sum_{s=2}^{m} k_{\nu l_2 \dots l_{s-1} \rho l_{s+1} \dots l_m} C_{j_1 l_s}^{\rho} \right] C_{j_2 j_3}^{l_2} \dots C_{j_{2m-2} j_{2m-1}}^{l_m} = 0 \quad . \end{aligned} \tag{12}$$

This result follows recalling

$$\epsilon_{i_1 \dots i_p i_{p+1} \dots i_n}^{j_1 \dots j_p j_{p+1} \dots j_n} \epsilon_{j_{p+1} \dots j_n}^{l_{p+1} \dots l_n} = (n-p)! \epsilon_{i_1 \dots i_p i_{p+1} \dots i_n}^{j_1 \dots j_p l_{p+1} \dots l_n} \tag{13}$$

in the second equality, using the invariance of k [eq. (6)] in the third one and the JI in the last equality for each of the $(m-1)$ terms in the bracket.

This may be seen without using coordinates; indeed (10) is expressed as

$$\tilde{\omega}(X_\rho, X_{j_2}, \dots, X_{j_{2m-2}}, X_\sigma) := k([X_\rho, X_{j_2}], \dots, [X_{j_{2m-3}}, X_{j_{2m-2}}], X_\sigma) \quad , \tag{14}$$

and the (2m-1)-form ω is obtained antisymmetrizing (14) as in (11) (cf. (4)). Hence

$$(L_{X_\nu} \tilde{\omega})(X_{i_1}, \dots, X_{i_{2m-1}}) = -\sum_{p=1}^{2m-1} \tilde{\omega}(X_{i_1}, \dots, [X_\nu, X_{i_p}], \dots, X_{i_{2m-1}})$$

$$
\begin{aligned}
&= -\sum_{s=1}^{m-1} k([X_{i_1}, X_{i_2}], \ldots, [[X_\nu, X_{i_{2s-1}}], X_{i_{2s}}] + [X_{i_{2s-1}}, [X_\nu, X_{i_{2s}}]], \ldots, \\
&[X_{i_{2m-3}}, X_{i_{2m-2}}], X_{i_{2m-1}}) - k([X_{i_1}, X_{i_2}], \ldots, [X_{i_{2m-3}}, X_{i_{2m-2}}], [X_\nu, X_{i_{2m-1}}]) \\
&= -\sum_{s=1}^{m-1} k([X_{i_1}, X_{i_2}], \ldots, [X_\nu, [X_{i_{2s-1}}, X_{i_{2s}}]], \ldots, [X_{i_{2m-3}}, X_{i_{2m-2}}], X_{i_{2m-1}}) \\
&-k([X_{i_1}, X_{i_2}], \ldots, [X_{i_{2m-3}}, X_{i_{2m-2}}], [X_\nu, X_{i_{2m-1}}]) \\
&= (L_{X_\nu} k)([X_{i_1}, X_{i_2}], \ldots, [X_{i_{2m-3}}, X_{i_{2m-2}}], X_{i_{2m-1}}) = 0 \quad ;
\end{aligned}
\tag{15}
$$

where the JI has been used in the third equality and (8) in the last, *q.e.d.*

c) The generalized Jacobi condition

Now we are ready to check that the tensor ω introduced above verifies a generalized Jacobi condition that extends eq. (1) to multibracket algebras.

Theorem Let $\mathcal{G}$ be a simple compact algebra, and let ω be the non-trivial Lie algebra $(2p+1)$-cocycle obtained from the associated p invariant symmetric tensor on $\mathcal{G}$. Then ω verifies the *generalized Jacobi condition* (GJC)

$$
\epsilon^{j_1 \ldots j_{4p-1}}_{i_1 \ldots i_{4p-1}} \omega_{\sigma j_1 \ldots j_{2p-1}}{}^{\rho}_{\cdot} \omega_{\rho j_{2p} \ldots j_{4p-1}} = 0 \quad . \tag{16}
$$

Proof: Using (11), (10) and (13), the *l.h.s.* of (16) is equal to

$$
\begin{aligned}
&-(2p-3)! \epsilon^{j_1 \ldots j_{4p-1}}_{i_1 \ldots i_{4p-1}} k_{l_1 \ldots l_p \sigma} C^{l_1}_{\rho j_1} \ldots C^{l_p}_{j_{2p-2} j_{2p-1}} \omega^{\rho}_{\cdot j_{2p} \ldots j_{4p-1}} \\
&= -(2p-3)! \epsilon^{j_1 \ldots j_{4p-1}}_{i_1 \ldots i_{4p-1}} k^{l_1}_{\cdot \ldots l_p \sigma} C^{l_2}_{j_2 j_3} \ldots C^{l_p}_{j_{2p-2} j_{2p-1}} C^{\rho}_{l_1 j_1} \omega_{\rho j_{2p} \ldots j_{4p-1}} = 0 \quad ,
\end{aligned}
\tag{17}
$$

where the invariance of ω (eq. (9)) has been used in the last equality, *q.e.d.*

d) Multibrackets and higher-order simple Lie algebras

Eq. (16) now allows us to define higher-order simple Lie algebras based on $\mathcal{G}$ using [3] the Lie algebra cocycles ω on $\mathcal{G}$ as generalized structure constants:

$$
[X_{i_1}, \ldots, X_{i_{2m-2}}] = \omega_{i_1 \ldots i_{2m-2}}{}^{\sigma}_{\cdot} X_\sigma \quad . \tag{18}
$$

The GJC (16) satisfied by the cocycles is necessary since for *even* n-brackets of associative operators one has the generalized Jacobi identity

$$
\frac{1}{(n-1)!n!} \sum_{\sigma \in S_{2n-1}} (-)^{\pi(\sigma)} [[X_{\sigma(1)}, \ldots, X_{\sigma(n)}], X_{\sigma(n+1)}, \ldots, X_{\sigma(2n-1)}] = 0 \quad . \tag{19}
$$

This establishes the link between the $\mathcal{G}$-based *even* multibracket algebras and the *odd* Lie algebra cohomology cocycles on $\mathcal{G}$ (note that for n odd the *l.h.s* is proportional to the odd ($2n$-1)-multibracket $[X_1, \ldots, X_{2n-1}]$ [3]).

Finally we comment that just in the same way that we can introduce for a Lie algebra a BRST nilpotent operator by

$$
s = -\frac{1}{2} c^i c^j C_{ij}{}^{k}_{\cdot} \frac{\partial}{\partial c^k} \equiv s_2 \quad , \quad s^2 = 0 \quad , \tag{20}
$$

with $c^ic^j=-c^jc^i$, the set of invariant forms ω associated with a simple $\mathcal{G}$ allows us to *complete* this operator in the form

$$
\begin{aligned}
s = & -\frac{1}{2}c^{j_1}c^{j_2}\omega_{j_1j_2\cdot}{}^{\sigma}\frac{\partial}{\partial c^\sigma} - \ldots - \frac{1}{(2m_i-2)!}c^{j_1}\ldots c^{j_{2m_i-2}}\omega_{j_1\ldots j_{2m_i-2}\cdot}{}^{\sigma}\frac{\partial}{\partial c^\sigma} - \ldots \\
& -\frac{1}{(2m_l-2)!}c^{j_1}\ldots c^{j_{2m_l-2}}\omega_{j_1\ldots j_{2m_l-2}\cdot}{}^{\sigma}\frac{\partial}{\partial c^\sigma} \equiv s_2 + \ldots + s_{2m_i-2} + \ldots + s_{2m_l-2}. \qquad (21)
\end{aligned}
$$

This new nilpotent operator s is the *complete BRST operator* [3] associated with $\mathcal{G}$.

For the relation of these constructions with the strongly homotopy algebras [6], possible extensions and connections with physics we refer to [3] and references therein.

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